

Unit 2: An Introduction to Partial Derivatives

1. Lecture 3.020

Calculus of Several Variables

Given $w = f(x_1, \dots, x_n)$, we let $n=2$ (+ to take advantage of geometry)
 \therefore let $w = f(x, y)$ where x and y are independent

Fix y at y_0 and let x vary between $x_0 - |\Delta x|$ and $x_0 + |\Delta x|$.

Def.
 $f_x(x_0, y_0) = \lim_{\Delta x \rightarrow 0} \left[\frac{f(x_0 + \Delta x, y_0) - f(x_0, y_0)}{\Delta x} \right]$

Similarly
 $f_y(x_0, y_0) = \lim_{\Delta y \rightarrow 0} \left[\frac{f(x_0, y_0 + \Delta y) - f(x_0, y_0)}{\Delta y} \right]$

In general, if $w = f(x_1, \dots, x_n)$ then
 $f_{x_i}(a_1, \dots, a_n) = \lim_{\Delta x_i \rightarrow 0} \left[\frac{f(a_1, \dots, a_i + \Delta x_i, \dots, a_n) - f(a_1, \dots, a_i, \dots, a_n)}{\Delta x_i} \right]$

Example:
 $w = f(x_1, x_2, x_3) = x_1 x_2 x_3 e^{x_1}$
 $f_{x_1}(x_1, x_2, x_3) = x_2 x_3 e^{x_1}$
 $f_{x_1}(1, 2, 3) = (2)(3)e^1 = 6e$
 $f_{x_2}(x_1, x_2, x_3) = x_1 x_3$
 $f_{x_2}(1, 2, 3) = (1)(3) = 3$

Often write $\frac{\partial w}{\partial x_i}$ instead of $f_{x_i}(x_1, \dots, x_n)$

$\frac{dy}{dx}$ versus $f'(x)$

a.

f_{x_1}, \dots, f_{x_n} called partial derivatives of f w.r.t x_1, \dots, x_n resp.

"usual" derivative properties still hold

Example.
 Let $w = e^{3x+y} \sin(2x-y)$
 Then:
 $\frac{\partial w}{\partial x} = e^{3x+y} [2 \cos(2x-y) + 3e^{3x+y} \sin(2x-y)]$

Caution: $\left(\frac{\partial u}{\partial x} \right)^{-1} \neq \left(\frac{\partial x}{\partial u} \right)$

$u = 2+y \rightarrow \frac{\partial u}{\partial x} = 1$
 $v = 2-y \rightarrow \frac{\partial v}{\partial x} = 0$
 $x = \frac{1}{2}u + \frac{1}{2}v \rightarrow \frac{\partial x}{\partial u} = \frac{1}{2}$
 $\therefore \frac{\partial u}{\partial x} \neq \left(\frac{\partial x}{\partial u} \right)^{-1}$

but -
 $u = x+y \rightarrow x = u-y$
 $\therefore \frac{\partial x}{\partial u} = 1$
 $\left(\frac{\partial u}{\partial x} \right)^{-1} = \left(\frac{\partial x}{\partial u} \right)$

Example (Explanation?)
 $w = e^{xy} \cos(z-y)$
 Let $u = z+y, v = z-y$
 $w = f(u, v) = e^{uv} \cos(v)$
 $= g(u, v) = e^{uv} \cos v$

Generally we consider $w = w(x, y)$ or $w = w(u, v)$

handle, even $w = w(u, y)$
 [think of polar vs Cartesian coordinates - never use, say, (r, θ)]

b.

Pictorially ($n=2$)

$w = f(x, y)$
 $P_0(x_0, y_0, w_0)$
 $x = x_0$
 $y = y_0$

$w = f(x, y)$
 $P_0(x_0, y_0, w_0)$
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 $y = y_0$

$\vec{N} = \vec{V}_1 \times \vec{V}_2$

$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 1 & \frac{\partial w}{\partial y} \\ 1 & 0 & \frac{\partial w}{\partial x} \end{vmatrix}$

$= \frac{\partial w}{\partial x} \vec{i} + \frac{\partial w}{\partial y} \vec{j} - \vec{k}$

Tangent Plane:
 $\frac{\partial w}{\partial x}(x-x_0) + \frac{\partial w}{\partial y}(y-y_0) - (w-w_0) = 0$

$\therefore \Delta w_{\text{tan}} = \frac{\partial w}{\partial x}(x_0) \Delta x + \frac{\partial w}{\partial y}(y_0) \Delta y$

c.

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2. Read Thomas, Sections 15.2 and 15.3.

3. Exercises:

3.2.1(L)

- a. If $f(x,y) = x^2 + y^3$, compute $f_x(1,2)$.
- b. If $f(x,y) = x^3y + x^4 + y^5$, compute $f_{xx}(1,2)$ where $f_{xx}(1,2)$ means $\frac{\partial}{\partial x} \left[\frac{\partial f}{\partial x} \right] (1,2)$.
- c. Find $f_y(1,2,3,4)$ if $f(w,x,y,z) = w^2xy + z^3y^2 + x^3zw$.

3.2.2

- a. Determine $f_w(w,x,y,z)$ and $f_{ww}(w,x,y,z)$ if $f(w,x,y,z) = w^3x^2y + x^3y^2z + wz^4$. In particular, determine $f_{ww}(1,2,3,4)$.
- b. Compute $\frac{\partial z}{\partial x}$ if $z^3xy + z^5y + \cos z = 1$.

3.2.3(L)

Let x and y be a pair of independent variables and define u and v by $u = 2x - 3y$ and $v = 3x - 4y$.

- a. Show that u and v are then also a pair of independent variables.
- b. Solve the above equations and express x and y in terms of u and v . From this compute $\frac{\partial x}{\partial u}$ and compare this with $\frac{\partial u}{\partial x}$.
- c. Express x in terms of u and y , and then compute $\frac{\partial x}{\partial u}$. How does this answer compare with the result in (b)?
- d. Express u in terms of x and v , and then compute $\frac{\partial u}{\partial x}$. In this case, does $\frac{\partial u}{\partial x} = \frac{1}{\left(\frac{\partial x}{\partial u}\right)}$ where $\frac{\partial x}{\partial u}$ is as in (b)?

3.2.4

Given that x and y are independent variables, define u and v by $u = x^2 - y^2$ and $v = 2xy$.

- a. Explain why u and y are also independent variables.

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3.2.4 continued

- b. Show that $\left(\frac{\partial x}{\partial u}\right)_y = \frac{1}{\left(\frac{\partial u}{\partial x}\right)_y}$.
- c. Determine the value of $\left(\frac{\partial x}{\partial u}\right)_v$.

3.2.5(L)

From the polar coordinate relations, $y = r \sin \theta$ and $x = r \cos \theta$, compute $\frac{\partial \theta}{\partial y}$ where, from now on, $\frac{\partial \theta}{\partial y}$ will be interpreted to mean $\left(\frac{\partial \theta}{\partial y}\right)_x$ unless otherwise specified.

3.2.6

Given that x and y are independent variables, assume that u and v are functions of x and y [we usually denote this as either $u = u(x,y)$ and $v = v(x,y)$, or, if we feel that misinterpretation might arise, as $u = f(x,y)$ and $v = g(x,y)$] such that u and v are also independent variables. Assume further that we also know that u and v are related by $u^2 = y^2 v$. Determine $\left(\frac{\partial u}{\partial y}\right)_x$.

3.2.7(L)

Let S be the surface defined by the Cartesian equation $z = x^2 + y^3$. Assume that there is a plane which is tangent to S at the point $P(1,2,9)$. Find the equation of this plane.

3.2.8

Assuming that the surface defined by the Cartesian equation $z = x^3 y^2 + x^5 + y^7$ has a tangent plane at the point $(1,1,3)$, find the equation of this plane.

3.2.9(L)

Let the surface S have the Cartesian equation $x = g(y,z)$.

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3.2.9(L) continued

- a. Assuming that S possesses a tangent plane at the point (x_0, y_0, z_0) , find the equation of this plane.
 - b. The plane M is tangent to the surface $x = e^{3y-z}$ at the point $(1, 2, 6)$. Find the equation of M .
 - c. Check the solution in (b) by expressing $x = e^{3y-z}$ in the form $z = f(x, y)$.
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