

Unit 2: Introduction to Matrix Algebra

1. Lecture 4.020

The Game of Matrices

S_n = set of all $n \times n$ matrices

$A \in S_n$ means

$$A = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nn} \end{bmatrix} = [a_{ij}]$$

Def. of Equal

$$[a_{ij}] = [b_{ij}] \iff a_{ij} = b_{ij}$$

Def. of Addition

$$[a_{ij}] + [b_{ij}] = [c_{ij}]$$

where $c_{ij} = a_{ij} + b_{ij}$
(term-by-term)

Def. of Mult

$$[a_{ij}] [b_{kj}] = [d_{ij}]$$

where

$$d_{ij} = \sum_{k=1}^n a_{ik} b_{kj}$$

Example #1

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$$

(1) $A \neq B$ ($a_{12} \neq b_{12}$)

(2) $A+B = \begin{bmatrix} 2 & 3 \\ 2 & 2 \end{bmatrix}$

(3) $AB = \begin{bmatrix} 2 & 3 \\ 2 & 3 \end{bmatrix}$

(4) $BA = \begin{bmatrix} 3 & 3 \\ 2 & 2 \end{bmatrix}$

a.

Properties (Rules) of the Game

- $A+B = B+A$
- $A+(B+C) = (A+B)+C$
- $I_0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$
then $A+I_0 = A$
- If $A = [a_{ij}]$
then $A+(-A) = I_0$
where $-A = [-a_{ij}]$

5. $A(BC) = (AB)C$

6. $A(B+C) = AB+AC$

7. If $I_n = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}$
then $[I_n] S_n = S_n$
 $AI_n = I_n A = A$

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

Note

$AI_n = I_n A$ is not redundant.

Differences from "usual" Algebra

(i) AB need not equal BA

(ii) A^{-1} need not exist - i.e., Given $A \neq 0$, the equation $AX = I_n$ need not be solvable

b.

Example #2

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

$$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$AX = \begin{bmatrix} x_1 + 2x_2 \\ 2x_1 + 4x_2 \end{bmatrix}$$

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$AX = I_2 \rightarrow$

$$\begin{cases} x_1 + 2x_2 = 1 \\ 2x_1 + 4x_2 = 0 \end{cases} \rightarrow 2=0$$

Must beware of results which require A^{-1}

E.g., we may have

(a) $AB=0$, yet $A, B \neq 0$
or

(b) $AB=AC$ yet $A \neq 0$ and $B \neq C$

Example #3

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 2 & 2 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 2 & -6 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} -2 & -2 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

c.

Lecture 4.020 continued

∴ Special interest in those A 's for which A^{-1} exists

Definition
 A is called non-singular $\Leftrightarrow A^{-1}$ exists

If A is non-singular, then
 $AB = AC \rightarrow B = C$
 i.e.
 $AB = AC \rightarrow A^{-1}(AB) = A^{-1}(AC) \rightarrow (A^{-1}A)B = (A^{-1}A)C \rightarrow I_n B = I_n C \rightarrow B = C$

Role of Determinants
 (More details in a later Block)
 A is non-singular $\Leftrightarrow \det A \neq 0$
 i.e.
 A^{-1} exists $\Leftrightarrow \det A \neq 0$

d.

Proof for $n=2$
 $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \rightarrow$
 $\begin{cases} ax_1 + by_1 = 1 \\ cx_1 + dy_1 = 0 \end{cases}$
 $\begin{cases} ax_2 + by_2 = 0 \\ cx_2 + dy_2 = 1 \end{cases}$
 Solvability requires $\frac{1}{a} \neq \frac{d}{c}$, or $ad - bc \neq 0$

Example #4
 If $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$
 $\det A = 4 - 4 = 0$
 $\therefore A$ is singular (i.e. non-non-singular)
 This checks with previous result that A^{-1} doesn't exist.

Example #5
 If $A = \begin{bmatrix} 5 & 7 \\ 3 & 6 \end{bmatrix}$
 then $\det A = 30 - 21 \neq 0$
 $\therefore A^{-1}$ exists
Major Problem (To be solved next lecture)
 Knowing that $\det A \neq 0$, how do we construct A^{-1} ?

e.

Study Guide
Block 4: Matrix Algebra
Unit 2: Introduction to Matrix Algebra

2. Read: Supplementary Notes, Chapter 6, Section D

3. (Optional) Read: Thomas, Section 13.2

4. Exercises

4.2.1

Show that $A(B + C) = AB + AC$ and $A(BC) = (AB)C$ where $A, B,$ and C are each 2×2 matrices.

4.2.2

Let $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ and $0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

- Find all matrices B such that $AB = 0$.
- Find all matrices C such that $AC = 0$ but $CA \neq 0$.

4.2.3

Let $A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ and $0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

Find all matrices B such that $AB = 0$

4.2.4

Let $A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ and $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

- Find a matrix X such that $AX = I$
- Does there exist a matrix X such that $BX = I$?

4.2.5

For any square matrix A , we define A^r to mean $\underbrace{AA \dots A}_r$, where r is any positive integer,

n times

- Compute A^3 if $A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$
- Compute A^x if $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

(continued on the next page)

4.2.5 (continued)

- c. Compute A^2 and A^3 if $A = \begin{pmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{pmatrix}$

4.2.6

Let $A = \begin{pmatrix} 5 & 3 \\ 3 & -3 \end{pmatrix}$ and let $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

- a. Compute $A^2 - 2A - 24I$
- b. Compute A^3 using the result of part (a).
- c. Compute A^7

4.2.7

Let $A = \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}$. Find all matrices B such that $AB = BA$.

4.2.8

If A is any matrix, we define the transpose of A , written A^T , to be the matrix obtained when we interchange the rows and columns of A . That is, the columns of A are the rows of A^T .

- a. What is A^T if $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$.
- b. If A is any matrix, compute $(A^T)^T$.
- c. If A and B are 2×2 matrices show that
- $$(AB)^T = B^T A^T.$$
- d. Find all 2×2 matrices A for which $AA^T = A^T A$.
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