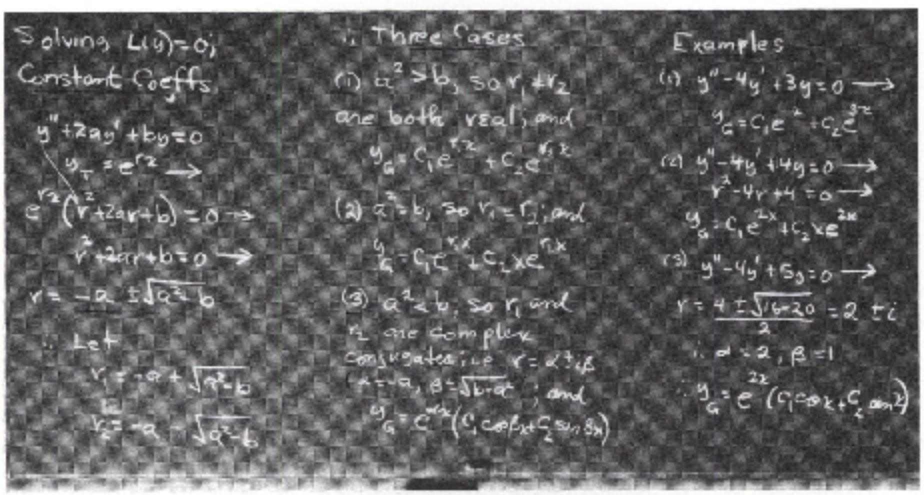
Unit 5: Linear Equations with Constant Coefficients

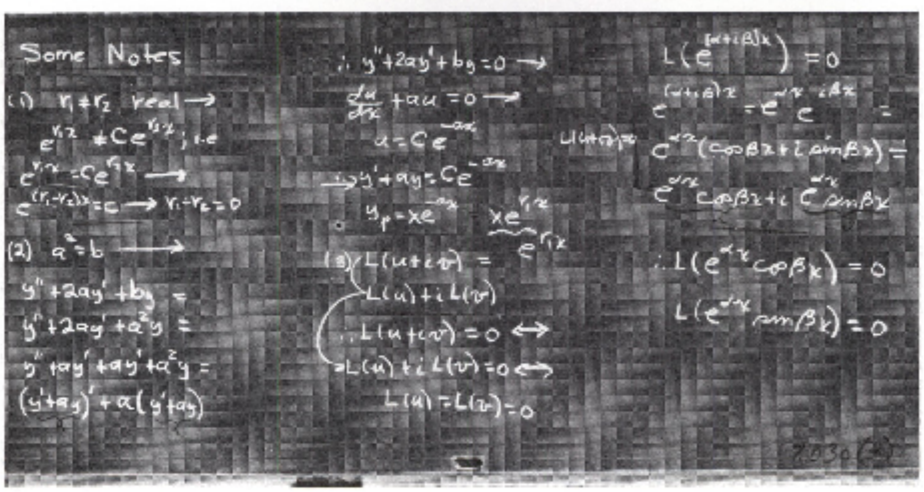
1. Overview

In the last unit, we gave an overview of how one obtains the general solution of a linear differential equation; and, in the exercises, we indicated how the special case of constant coefficients facilitated computational techniques. In this unit, we investigate the problem of obtaining the general solution of the homogeneous (reduced) linear equation L(y) = 0 in the case that L(y) has constant coefficients.

2. Lecture 2.030



a.



b.

3. Read Thomas, Sections 20.8 and 20.9. [In trying to understand these sections, it is not important that you be able to understand the use of differential operators (such as D, (D - 1)y, etc.) mentioned in the text. We have expanded the notion of differential operators as an optional topic for the time being (Chapter 4 of the Supplementary Notes) but which fits in very nicely with the material on linearity in Block 3. The interested reader is free to read Chapter 4 if he so wishes, but no knowledge of differential operators is necessary in the remainder of this block.]

4. Do Exercises 2.5.1 through 2.5.2 (or 2.5.3).

5. Read Supplementary Notes, Chapter 3, Section C.

6. Do the rest of the exercises.

7. Exercises:

2.5.1

Find the general solution of each of the following equations:

a. $y'' - 9y' - 36y = 0$.

b. $y'' - 12y' + 36y = 0$.

c. $y'' - 8y' + 25y = 0$.

2.5.2(L)

a. Find the general solution of

$$L(y) = 0,$$

where $L(y) = y'' + 2y' - 3y$ by computing $L(e^{rx})$ and then equating this to zero.

b. Assuming that

$$\frac{\partial L(e^{rx})}{\partial r} = L\left[\frac{\partial}{\partial r}(e^{rx})\right],$$

find the general solution of $L(y) = y'' - 14y' + 49 = 0$.

2.5.3 (Optional)

[This exercise reinforces the note at the end of the solution of Exercise 2.5.2. For the student who does not want to pursue this point further, Exercise 2.5.3 may be omitted since it has no bearing on the remaining exercises.]

$P_5(r)$ is a fifth-degree polynomial in r with real coefficients. It is known that $r_1 = 3$, $r_2 = 4 + 3i$, and $r_3 = 5 - 2i$ are roots of the polynomial equation, $P_5(r) = 0$. Determine $P_5(r)$ if we assume that its leading coefficient is 1.

2.5.4

Find the general solution of

$$\frac{d^5y}{dx^5} - 2\frac{d^3y}{dx^3} + \frac{dy}{dx} = 0.$$

2.5.5

Find the curve c which satisfies $\frac{d^4y}{dx^4} - 2\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2} = 0$ such that c passes through (0,2) in such a way that at this point $\frac{dy}{dx} = 0$, $\frac{d^2y}{dx^2} = 2$, and $\frac{d^3y}{dx^3} = 0$.

2.5.6(L)

Find the general solution of

$$y''' - 3y'' + 3y' - y = 0.$$

2.5.7 (Optional)

[The primary role of this exercise is to reinforce the note at the end of Exercise 2.5.6(L). Otherwise, it is not essential to our general discussion.]

Find the lowest order linear differential equation with constant coefficients, $L(y) = 0$, for which $y = xe^{2x}\cos 3x$ is a particular solution.

2.5.8

Find the general solution of

$$\frac{d^6y}{dx^6} - 2\frac{d^3y}{dx^3} + y = 0.$$

2.5.9 (Optional)

[The role of this exercise is to establish the result that if L(y) has constant coefficients,

$$\frac{\partial L(e^{rx})}{\partial r} = L\left[\frac{\partial (e^{rx})}{\partial r}\right] = L(xe^{rx}).$$

The student who is willing to accept this result on faith alone may omit this exercise.]

a. Compute

$$(xe^{rx})', \ (xe^{rx})'', \text{ and } (xe^{rx})'''$$

where the differentiation is with respect to x and r is being treated as a constant.

b. Use the result of (a) to conjecture that

$$\frac{d^k(xe^{rk})}{dx^k} = (kr^{k-1} + xr^k)e^{rx}$$

and verify by induction that this conjecture is correct.

c. Use (b) to establish that $\frac{\partial L(e^{rx})}{\partial r} = L(xe^{rx})$ in the special case that $L(y) = \frac{d^ky}{dx^k}$.