Unedited Draft

Arithmetic Revisited

Lesson 2 :

The Arithmetic of Common Fractions

Part 7: Appendix to Lesson 3

Some Elementary Number Theory

Part 1: Prime Numbers and GCF's

1. Introduction to Prime Numbers:

Listed below are the first few multiples of 6.

6, 12, 18, 24, 30, 36, 42, 48

Notice that neither 4 nor 9 is a member of this list.

However, 4×9, their product, is on the list.

Why do things like this happen? To begin with, notice that $6 = 3 \times 2$; and since 6 is a multiple of 2 and 3, any multiple of 6 is also divisible by both 2 and 3.

With this in mind if we look more closely at 4 and 9, we see that $4 = 2 \times 2$ and $9 = 3 \times 3$. Hence we may rewrite 4×9 in the following way:

$$4 \times 9 = (2 \times 2) \times (3 \times 3) = (2 \times 3) \times (2 \times 3)$$

In other words, even though neither 4 nor 9 were on the list of multiples of 6, the 2 (which is a factor[1] of 4) and the 3 (which is a factor of 9) "bonded" to form a factor of 6.

Now let's revisit the same question but this time with reference to the multiples of 5. The first few multiples of 5 are:

5, 10, 15, 20, 25, 30, 35, 40

And again we see that neither 4 nor 9 are members of this list. This time, however, neither is 4×9,

[1]We use the words "divisor" and "factor" as meaning the same thing. That is, if 6 is divisible by 2 we may say either that 2 is a divisor of 6 or 2 is a factor of 6.

What happened this time that did not happen last time? The answer is that other than for itself and 1, 5 has no divisors. Hence it has to be "used up" either all at once or not at all, This is unlike what happened when we were dealing with 6, in which case 6 had 2 and 3 as factors; and the 2 could be "used up" with the 4 and the 3 could be "used up" with the 9.

In summary, the important thing from our point of view is that if the product of two whole numbers is divisible by 5 at least one of the two numbers must itself be divisible by 5. However if the product of two numbers is divisible by 6 we can not conclude for sure that at least one of the two numbers is divisible by 6 (as we just saw),

This subtle but important difference between a number being divisible by 6 and a number being divisible by 5 leads to an important definition, namely:

<u>Definitions</u>:

- **A whole number, greater than 1, is called a prime number if and only if its only divisors are 1 and itself.**
- **1 is called a unit.**
- **All other whole numbers are called composite numbers.[2]**

Let's make sure we have internalized the difference between a prime number and a composite number by looking at one more example. Listed below are the first ten multiples of 15.

15, 30, 45, 60, 75, 90, 105, 120, 135, 150

Notice that neither 6 nor 20 are on this list but that 6×20 is on the list.

Let's look into why this happens.

-- Any composite number can be written as a product of prime numbers. The idea is that if a number is not prime it must have at least two factors, neither of which is 1. Each of these two factors is either a prime number or composite. If it's composite we repeat the procedure and eventually, we will be left with a product of prime numbers.

- **In the case of 6, it can be written as 2×3; and both 2 and 3 are prime numbers**
- **In the case of 20 it can be written as a product in several ways. one such way is as 2×10. In this case 2 is a prime number but 10 isn't. However 10 can be rewritten as 2×5. Hence:**

$$20 = 2 \times 10 = 2 \times (2 \times 5) = 2 \times 2 \times 5$$

and both 2 and 5 are prime numbers.

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[2]Every whole number is a product of 1 and itself. The difference is that if the number is prime these are its only factors.

Thus we may rewrite the product 120 as:

$$120 = 6 \times 20 = (2 \times 3) \times (2 \times 2 \times 5) = (2 \times 2 \times 2) \times (3 \times 5) = 8 \times 15$$

and from this we see not only that 120 is a multiple of 15 but that, in particular, it is the 8th multiple of 15.

2. The Prime Factorization Theorem

In arriving at the prime factors of 120 we started by rewriting 120 as 6×20. The point is that it would have been just as logical to have started with rewriting 120 as 10×12, in which case neither 10 nor 12 are prime numbers. However we can rewrite 10 as 2×5 and see that both 2 and 5 are prime numbers. We could rewrite 12 as, say 2×6, in which case 2 is a prime number but 6 isn't be. However we could then have rewritten 6 as 2×3, in which case both 2 and 3 are prime numbers. In this way we obtain:

$$120 = (2 \times 5) \times (2 \times 2 \times 3) =$$

Rearranging the above prime factors of 120 we again obtain the result:

$$120 = 120 = 2 \times 2 \times 2 \times 2 \times 3 \times 5.$$

The above result is not a coincidence. Except for the order in which we write the factors there is one and only one way to decompose a composite number into a product of prime numbers. To see why this is the case, suppose that p is a prime number that is a divisor of the product $2 \times 2 \times 2 \times 2 \times 3 \times 5$. Since p is a prime number the only way that it can be a divisor of a product is to be a divisor of at least one of the factors of the product. Thus p must be a divisor of either 2, 3 or 5. However since 2, 3 and 5 are themselves prime numbers they are divisible only by 1 (which, by definition, is not a prime number) or themselves. This means that p must be either 2, 3 or 5.

Conversely, if 120 is decomposed into a different product of prime numbers, it means that 2, 3, or 5 each have to divide one of those prime numbers. Hence each of the “new” prime factors must also be either 2, 3, or 5. In other words, every prime factor in one factorization is a prime factor in the other and vice versa. Thus except for the order in which the factors are written, the decomposition into a product of prime factors is unique.

This result can be generalized as follows:

> ***The Prime Factorization Theorem*:**
>
> **Every whole number greater than 1 can be written as a product of prime numbers; and except for the order in which the factors are written, this can be done in one and only one way.**
>
> **This result is also known as the *Fundamental Theorem of Arithmetic*.[3]**

Let's practice this idea one more time by decomposing 60 into a product of prime numbers in several different ways. To this end, let's first look at the number of different ways in which 60 can be written as a product of whole numbers. Namely:

$$\begin{aligned} 60 &= 1 \times 60 \\ &= 2 \times 30 \\ &= 3 \times 20 \\ &= 4 \times 15 \\ &= 5 \times 12 \\ &= 6 \times 10 \\ &= 3 \times 4 \times 5 \\ &= 2 \times 3 \times 10 \\ &= 2 \times 2 \times 3 \times 5 \end{aligned}$$

Of the above factorizations of 60, only the last one consists solely of prime numbers. However, if we further "reduce" the factors that aren't prime numbers, we see that each of the other factorizations can also be written as a product of prime factors. For example

$$\begin{aligned} 60 &= 6 \times 10 \\ 6 &= 3 \times 2 \\ 10 &= 5 \times 2 \end{aligned}$$

Therefore::

$$\begin{aligned} 60 &= (2 \times 3) \times (5 \times 2) \\ &= 2 \times 3 \times 5 \times 2 \\ &= 2 \times 2 \times 3 \times 5 \end{aligned}$$

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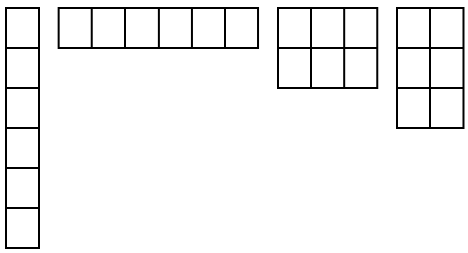
[3]Many times people wonder why we prove things that may seem to be intuitively obvious. The answer is that intuition is not always correct. By being able to supply a rigorous proof of a statement whose truth may be doubted, we show that even if it's not "self-evident" the statement is still true.

A Possible Classroom Activity:

Give students a set of square tiles, all of the same size and ask them to make rectangular windows in as many ways as possible. For example if the student has 6 tiles the possible arrangements of the tiles are:



Notice in this case that even though $3 \times 2 = 2 \times 3$, a window that has two rows each with three panes is not the same shape as a window that has three rows each with two panes.

Let the students discover that with one tile there is only one shape and that if there are a prime number of tiles there can only be two shapes (for example, with 7 tiles the window can either be 7 by 1 or 1 by 7). All other numbers of tiles will form more than 2 shapes. In fact the number of shapes will always be an odd number unless the number is a perfect square (can you see why?)

Such an activity might help children better visualize what prime numbers are.

3. The Greatest Common Divisor (gcf):

Prior to the advent of the hand held calculator it was a good strategy to "reduce fractions to lowest terms". This concept was based on the fact that if you divided (or multiplied) numerator and denominator of a fraction by the same non-zero number you obtained an equivalent fraction. When you obtained an equivalent fraction in which the only common divisor of the numerator and denominator was 1, the fraction was said to be in lowest terms.

Illustrative Example:

Reduce $\frac{18}{24}$ to lowest terms.

***Solution*:**

Since $\frac{18}{24}$ represents the ratio $18 \div 24$ and since we know that we do not change the ratio be dividing both numbers in the ratio by the same (non-zero) number, we may "cancel" 2 from both the numerator and denominator to obtain the equivalent fraction $\frac{9}{12}$.

Both the numerator and denominator of $\frac{9}{12}$ are divisible by 3. Hence we may cancel the common factor 3 from both the numerator and denominator of $\frac{9}{12}$ to obtain the equivalent fraction $\frac{3}{4}$.

Notes:

- **Although 4 is not a prime number, it has no factor (other than 1)[4] in common with 3. Thus in a sense, 4 is a prime number relative to 3. More formally:**

> **Definition:**
>
> **The whole numbers m and n are said to be relatively prime if their only common factor is 1.**
>
> **Thus, for example, even though neither 8 nor 9 are prime numbers they are a pair of relatively prime numbers (because their only common factor is 1.**
>
> **In the language of "relatively prime" a fraction is said to be in lowest terms if and only if its numerator and denominator are relatively prime.**

- **In terms of a real-life illustration saying that you can buy 18 pens for $24 is equivalent to saying you can buy 3 pens for $4. Prior to the advent of calculators it was much less tedious to deal with 3 and 4 rather than with 18 and 24 when doing calculations.**

- **The fact that both 2 and 3 are common factors of 18 and 24 means that 6 (that is, 2×3) is also a common factor of 18 and 24.[5]**

- **In fact, since 3 and 4 are relatively prime, 6 is not merely a common factor of 18 and 24. Rather it is the greatest common factor of 18 and 24.**

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[4] Keep in mind that 1 is a divisor of every whole number.

[5] The truth of this statement follows because both 2 and 3 are prime numbers. Other wise it need not be true. For example, both 4 and 6 are factors of 12 but their product (24) is not a divisor of 12. In fact in this case it is a multiple of 12.

• In helping students keep track of the arithmetic when they are reducing fractions to lowest terms, some textbooks used a special notation for indicating that both the numerator and denominator of $\frac{18}{24}$ were divisible by 2. For example, they might have written this in the form:

$$\begin{array}{r|l} 2 & 18 \\ \hline 2 & 24 \end{array}$$

• The above notation helps us internalize a notation that if often used by mathematicians. Namely to indicate that 2 is a factor of 18 they would write $2|18$. So rather than take the time to write such things as "4 is a divisor of 36" or, equivalently, "36 is a multiple of 4" we will simply write $4|36$. More generally:

Definition:

• If m and n are whole numbers and m is a factor of n (or, equivalently, if n is a multiple of m) we abbreviate this fact by writing $m|n$.

• We call d the greatest common factor of b and c if $d|b$ and $b|c$ but no number greater than d is a factor of both b and c. We usually write gcf as an abbreviation for "greatest common factor". Thus, for example, we would write gcf (18, 24) = 6 to indicate that 6 was the greatest common factor of 18 and 24.

• While gcf (18, 24) = 6 notice that gcf (12, 24) = 12, not 6. That is, while 6 is a factor of both 12 and 24 it is not the greatest common factor

To generalize the above discussion, the point is that no matter what common fraction we were given we could eventually replace it by an equivalent common fraction in which the numerator and denominator were relatively prime. Namely, we keep dividing the numerator and denominator by common factors until we obtain an equivalent common fraction in which the numerator and denominator are relatively prime. No matter how complicated the numerator and denominator might be, this process, although it might be tedious at times, will always work.

Practice Problem # 1:

Reduce $\frac{48}{72}$ to lowest terms.

Answer: $\frac{2}{3}$

Solution:

If we were comfortable with our "arithmetic facts" we might notice that both 48 and 72 are divisible by 24.[6] And if we divide both 24 and 72 by 24 we obtain the equivalent common fraction $\frac{2}{3}$. Because 2 and 3 are both prime numbers they are automatically relatively prime. Therefore gcf (24, 72) $=$ 12 and $\frac{2}{3}$ is in lowest terms.

Notes:

- **Notice that every common factor of 24 and 72 is also a factor of the greatest common factor of 24 and 72. That is, 1, 2, 3, 4, 6 and 12 are all factors of 12**

- **We did not have to recognize that 12 was a common factor of 24 and 72. Rather we could have first noticed that both 24 and 72 are divisible by 2 and therefore we could cancel a factor of 2 from the numerator and denominator to obtain the equivalent fraction $\frac{12}{36}$. Then since 12 and 36 are also both even, we can divide the numerator and denominator by 2 again to obtain $\frac{6}{9}$. 2 is no longer a common factor of both the numerator and denominator but 3 is. Hence we may divide numerator and denominator by 3 to obtain $\frac{2}{3}$.**

It is relatively easy to find the factors of 48 and 72. However it is not as easy to find the factors of, say, 1,189. However when we reduce a fraction to lowest terms we can only "cancel" common factors; and this can sometimes be helpful when we deal with fractions that contain a numerator or denominator such as 1,189. For example:

Practice Problem #2:

Reduce $\frac{58}{1{,}189}$ to lowest terms.

Answer: $\frac{2}{41}$

[6]In terms of our new notation we could write $24|48$ and $24|72$.

Solution

• As we just mentioned, it is not east to find the factors of 1,189. However the fact that 58 is an even number tells us that 2 is a factor of 58. Hence the fraction can be rewritten as:

$$\frac{2 \times 29}{1{,}189}$$

Since 29 is a prime number we know that the only two prime factors of 58 are 2 and 29. Moreover, because 1,189 ends in an odd digit, we know that 2 is not a factor of 1,189. So the only possible way that $\frac{58}{1{,}189}$ can be reduced is if 29 is also a divisor of 1,189; and a quick check shows that $1{,}189 \div 29 = 41$. Hence:

$$\frac{58}{1{,}189} = \frac{2 \times 29}{1{,}189} = \frac{2 \times 29}{41 \times 29}$$

and we may then "cancel" 39 from the numerator and denominator of $\frac{2 \times 29}{41 \times 29}$ to obtain $\frac{2}{41}$. Since 2 and 41 are relatively prime (in fact they are both prime numbers), the fraction is now in lowest terms.

Notice how much more convenient it is to think in terms of "2 for $41" than in terms of "58 for $1,189".

The above problem illustrates the importance of numbers being relatively prime when we are reducing fractions to lowest terms. However it was a stroke of good luck in this problem that it was easy to find the factors of one of the two numbers in the fraction. A much more computationally complicated problem would have been to reduce a fraction such as

$$\frac{1{,}853}{3{,}379}$$

to lowest terms.

It turns out that the greatest common factor of 1,853 and 3,379 is 109. In fact:

$$1{,}853 = 109 \times 17 \text{ and } 3{,}379 = 109 \times 31$$

However, a good question might be "Is there an efficient way for determining that 109 is the greatest common factor of 1,853 and 3,379 without enduring cumbersome amount of trial-and-error experimentation?"

A rather efficient procedure for finding the greatest common factor of any pair of whole numbers is known as the Euclidean algorithm and it will be discussed in more detail soon.[7]

[7]The derivation of the Euclidean algorithm may be beyond the scope of an elementary arithmetic course but the algorithm itself is a nice way to have students practice their whole number arithmetic skills.

4. GCF's and the Euclidean Algorithm

The name "Euclidean algorithm" is probably more threatening than the concept it represents. Namely, given any two whole numbers we can always divided the greater one by the lesser one and obtain a remainder that is less than the smaller number. For example, given the numbers 24 and 90, we can divide 90 by 24 to obtain 3 and a reminder of 18. In terms of the usual long division algorithm we see that:

$$\begin{array}{r|l} & \ 3 \quad \text{R } 18 \\ 24 & 9\ \ 0 \\ - & 7\ \ 2 \\ \hline & 1\ \ 8 \end{array}$$

In other words, we can write that:

$$90 = 3(24) + 18$$

The interesting thing is that the gcf of the divisor (24) and the remainder 18) have the same gcf as the original two numbers (90 and 24).

To see why this is true, we should first digress and talk a little about some of the properties of divisibility. Rather than to be too abstract, let's look at some of these properties in terms of some specific numbers and then generalize them. For example:

- The greatest common factor of 0 and 7 is 7. Namely, 0 is a multiple of everty non zero number (for example, $7 \times 0 = 0$) and the grweaterst number that is a divsor of 7 is 7 itself. Hence gcf $(7,0) = 0$/ More generally:

If n is any non zero whole number then gcf(n,0) $=$ n

- Since 3|12, 3 also divides any multiple of 12.

Recall that 3|12 means that 3 is a divisor of 12; or, equivalently, 12 is a multiple of 3. More specifically, $12 = 4 \times 3$. Let's look at a specific multiple of 12. For example, $60 = 5 \times 12$. We may rewrite 12 in the form 4×3 and see that $60 = 5 \times (4 \times 3) = (5 \times 4) \times 3 = 20 \times 3$. This shows us not only that 3 is a factor of 60 but also that 60 is the 20th multiple of 2.

To see this in a more visual way, let's use a version of the corn bread we may represent 12 in the form:

3	3	3	3

and 60 in the form

3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3

Note:

To generalize this result simply replace 3 by d and 12 by n to obtain the more general result that if $d|n$ then d is a divisor of any multiple of n. [8] **In particular, if $d = gcf(m,n)$ then it is also true that $d = gcf(m, kn)$ where k is a whole number**

ince $3|18$ and $3|12$, then 3 is also a divisor of $18 + 12$ and $18 - 12$.

Again thinking in terms of our corn bread model, we may represent 12 in the form.

3	3	3	3

and 18 in the form

3	3	3	3	3	3

Then we may represent $18 + 12$ by the region shown below:

3	3	3	3	3	3	3	3	3	3

and $18 - 12$ is represented below by the region shown below

3	3	3	3	3	3
3	3	3	3	3	3

Note:

To generalize this result simply replace 3 by d, 12 by n and 18 by m to obtain the more general result that if $d|m$ and $d|n$ then $d|(m + n)$ and $d|(m - n)$. In particular if $d = gcf(m,n)$ then $d = gcf(m, m + n)$ and $d = gcf(m, m - n)$

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[8]The mathematical way of writing that "d is a divisor of every multiple of n" is " $d|kn$, where k is any whole number".

The above results have an interesting property when applied to the gcd of a pair of numbers such as 24 and 90. In terms of the Euclidean Algorithm w eknow that

$$90 = 3(24) + 18$$

We shall show that in terms of the long division jargon, the greatest common factor of the dividend (in this case, 90) and the divisor (in this case, 24) is equal to the greatest common factor of the divisor (24) and the remainder (n this case, 18).

Since we already know that gcf (90,24) = 6, it is sufficient to show that gcf(24,18) = 6 as well.

Well, we know that since 6|90 and 6|24, 6 must also be a factor of 3(24). And if 6|3(24) and 6|90 then it is also true that 6|18 because 18 = 90 − 3(24). since 6|24 and 6|18, 6 is a common factor of 24 and 18.

So let's suppose that d is the greatest common factor of 24 and 18. At the very least d has to be at least as great as 6 because d is the greatest common factor of 24 and 18 which means that any common factor of 24 and 18 must be a factor of d.

On the other hand since 90 = 3(24) + 18, the fact that d|18 and d|24 means that it is also true that d|90. Hence d is a divisor of both 24 and 90 and hence must be a factor of the greatest common factor of 24 and 90 (which is 6). Hence d|6 but d can't be less than 6. Hence d = 6. In summary,

gcf(90,24) = 6 = gcf(24,18)

There was nothing special about the choice of 90 and 24. More generally,our result can be sstatged as follows

In the language of the Euclidean Algorithm:

gcf (dividend, divisor) = gcf (divisor, remainder)

And repeated use of this fact allows us to find the greatest common factor of any two numbers.

While we most likely wouldn't use the Euclidean algorithm to find the greatest common divisor of 90 and 24, we might be very tempted to use it if we had been asked to reduce $\frac{1,517}{1,591}$ to lowest terms. In this case common factors of the numerator and denominator do not seem to “jump out” at us.

However using the euclidean Algorithm repeatedly, we see that:

$$1{,}591 = 1(1{,}517) + 74$$
$$1{,}517 = 20(74) + 37$$
$$74 \quad = 2(37) + 0$$

and our string of equalities becomes:

$$\text{gcf}(1591,\ 1517) = \text{gcf}(1517,\ 74) = \text{gcf}(74{,}37) = \text{gcf}(37{,}0) = 37$$

As a check we see that 1,591 $\div$ 37.43 and 1,517 $\div$ 37 $=$ 41. Therefore:

$$\frac{1{,}517}{1{,}591} = \frac{37\times41}{37\times43} = \frac{41}{43}$$

and sinced 41 and 43 are bothprime numbers and hence realtively prime, we know that $\frac{41}{43}$ is in lowest terms.[9]

Let's now return the question of how we might reduce $\frac{1{,}853}{3{,}379}$ to lowest terms. We again beign by using the Euclidean Algorithm repeatedly to obtain:

$$3{,}379 = 1(1{,}853) + 1{,}526$$
$$1{,}853 = 1(1{,}526) + 327$$
$$1{,}526 = 4(327) + 218$$
$$327 = 1(218) + 109$$
$$218 = 2(109) + 0$$

This leads to the string of equalities:

gcf (3,379, 1,853) $=$

gcf (1,853, 1,526) $=$

gcf (1,526, 327) $=$

gcf (327, 218) $=$

gcf (218,109) $=$

gcf (109,0) $=$

109

[9] Again to make sure we don't lose sight of the forest because of the trees, what this says is, for example, that buying items at a rate of 1,517 per $1,591 is the same rate as buying the items at a cost of $3 for every 41 items. In summary, 1,517 per 1,591 is the same rater as 41 per 43.

And if we now divide each of the numbers 3,379 and 1,853 by 109 we see that:

$$1{,}853 = 109 \times 17 \text{ and } 3{,}379 = 109 \times 31$$

Hence we may divide both numerator and denominator of $\frac{1,853}{3,379}$ by 109 to obtain the equivalent common fraction $\frac{17}{31}$. And since 17 and 31 are both prime numbers $\frac{17}{31}$ is in lowest terms.

5. Summary

In dealing with fractions at the elementary and middle school level we will rarely if ever be trying to reduce fractions of the type $\frac{1,853}{3,379}$. In general we will be dealing with numbers whose prime factors will usually be 2, 3, 5, 7 and/or 11 (and occasionally a few greater prime numbers). In such cases it usually means that the fractions can be reduced to lowest terms by dividing both numerator and denominator by the same prime number. For example, to reduce $\frac{12}{18}$ to lowest terms we may first divide numerator and denominator by 2 to obtain the equivalent common fraction $\frac{6}{9}$; and we can then divide 6 and 9 by 3 to obtain $\frac{2}{3}$.

We most likely would not have used the Euclidean algorithm (although we could have) to determine that the gcf of 12 and 18 is 6.

On the other hand the Euclidean algorithm makes it relatively easy for us to reduce a common fraction such as $\frac{4,699}{9,017}$ to lowest terms. More specifically:

$$9{,}017 = 1(4{,}699) + 4{,}318$$

$$4699 = 1(4{,}318) + 381$$

$$4{,}318 = 11(381) + 127$$

$$381 = 3(127).$$

Thus gcf (4,699, 9,017) = 127.

More specifically: $9{,}017 = 127 \times 71$ and $4{,}699 = 127 \times 37$. Hence we may divide both the numerator and the denominator of $\frac{4,699}{9,017}$ by 127 to obtain the equivalent common fraction $\frac{37}{71}$ by , which is in lowest terms (that is 37 and 71 are relatively prime).

Although students might not encounter problems such as this, it provides good practice in arithmetic for them to use the Euclidean algorithm to find common factors of two numbers when other methods prove to be too cumbersome.