Unedited Draft

Arithmetic Revisited

Lesson 4:

Part 2: Adding and Subtracting Mixed Numbers

1. Adding Mixed Numbers

As we saw in a previous illustration, if 38 cornbreads are divided equally among 7 people the number of cornbreads each person gets is $5 + \frac{3}{7}$. This might tempt us to write the answer as $5 + \frac{3}{7}$ cornbreads. The fact that the plus sign separates the whole number from the fraction makes it easy for one to assume that only the fraction is modifying "cornbreads"[1]. To avoid the possibility of misinterpretation, we should use parentheses and write $(5 + \frac{3}{7})$ cornbreads.

However because it's cumbersome to write the answer in this form, we agree to omit the plus sign and write the fractional part immediately to the right of the whole number; that is $5\frac{3}{7}$ means $(5 + \frac{3}{7})$.

In any event, this shows us that to add two mixed numbers we simply have to add the two whole numbers to get the whole number part of the sum and the two fractions to get the fraction part of the sum. More specifically:

Practice Problem #1

Express the sum of $5\frac{2}{7} + 6\frac{4}{7}$ as a mixed number.

Answer: $11\frac{6}{7}$

Solution

We may rewrite the problem in the form:

$$(5 + \tfrac{2}{7}) + (6 + \tfrac{4}{7})$$

Using the associative and commutative properties of addition we may rewrite the above expression (1) in the form:

$$(5 + 6) + (\tfrac{2}{7} + \tfrac{4}{7})$$

[1] This type of misinterpretation happens frequently to beginning students in algebra. For example, when they are trying to simplify an expression such as $(a + b)x$, they rewrite the expression as $a + bx$ rather than as $ax + bx$.

From this expression we see the sum is $11+\frac{6}{7}$; or in more standard form $11\frac{6}{7}$;

Warning: Looks Can Be Deceiving:

Once we've proceeded as we did above, it's okay to say that it was "only natural" when we add mixed numbers we add the whole numbers and we add the fractions. However, "natural" or not, it's a dangerous statement. After all, we saw an example of this type of "natural" thinking in the statement:

> *"When we add two common fractions, it's only natural to add the two numerators to get the numerator of the sum and to add the two denominators to get the denominator of the sum".*

In short, natural or not, the rule was incorrect for finding the sum of two rational numbers.

In fact the same type of logic might say

> *"It's only natural that when we multiply two mixed numbers to multiply the whole numbers and to multiply the fractions. Thus for example, $4\frac{1}{2}\times 2\frac{1}{2}=8\frac{1}{2}$"*

All one has to do to see that the above statement is false is to observe that $4\frac{1}{2}\times 2=9$ and that $4\frac{1}{2}\times 2\frac{1}{2}>4\frac{1}{2}\times 2$ We'll discuss this in more detail in a later section

• If we were uncomfortable with mixed numbers but were more comfortable with common fractions, we could translate every mixed number problem into an improper fraction problem. Conversely,if we prefer, we can convert improper fractions into mixed numbers. For example, we may rewrite $5\frac{2}{7}$ and $6\frac{4}{7}$ as:

$$\begin{aligned}5\tfrac{2}{7} &= 5+\tfrac{2}{7}\\ &= \tfrac{5}{1}+\tfrac{2}{7}\\ &= \tfrac{35}{7}+\tfrac{2}{7}\\ &= \tfrac{37}{7}\end{aligned}\qquad \text{and} \qquad \begin{aligned}6\tfrac{4}{7} &= 6+\tfrac{4}{7}\\ &= \tfrac{6}{1}+\tfrac{4}{7}\\ &= \tfrac{42}{7}+\tfrac{4}{7}\\ &= \tfrac{46}{7}\end{aligned}$$

Hence $5\frac{2}{7}\ +\ 6\frac{4}{7}=\frac{37}{7}+\frac{46}{7}=\frac{83}{7}=11\frac{6}{7}$.

However in this case it was probably much simpler just to add the whole numbers and to add the fractional parts.

Converting between Mixed Numbers and Common Fractions:

To convert a mixed number (such as $7\frac{4}{11}$) to a common fraction:

Multiply the whole number by the denominator ($7 \times 11 = 77$)

Add the numerator ($77 + 4 = 81$)

Then write the sum over the denominator ($\frac{81}{11}$)

In terms of our "cornbread" analogy, if each cornbread consists of 11 pieces then 7 cornbreads consist of 7×11 or 77 pieces.

$\frac{4}{11}$ of the cornbread is 4 pieces.

Hence, all in all we have $77 + 4$ or 81 pieces; and since each piece is $\frac{1}{11}$ of a cornbread, altogether we have 81 elevenths of a cornbread.

-- To convert an improper fraction (such as $\frac{81}{11}$) to a mixed number

Divide the numerator by the denominator ($81 \div 11 = 7$ R4)

Write the remainder over the denominator ($7\frac{4}{11}$). [2]

Practice Problem #2

Express the sum $6\frac{2}{3} + 3\frac{3}{4}$ as a mixed number.

Answer: $10\frac{5}{12}$

Solution:

We simply have to add the whole numbers and add the fractions. More specifically:

[2] Again in terms of our corn bread analogy, if there are 81 corn breads and 11 people, each person gets 7 corn breads and the remaining 4 corn breads are then divided equally between the 11 people, so that each person gets an additional $\frac{4}{11}$ of a corn bread.

$$
\begin{aligned}
6\tfrac{2}{3} + 3\tfrac{3}{4} &= (6+3) + (\tfrac{2}{3} + \tfrac{3}{4}) \\
&= (6+3) + (\tfrac{8}{12} + \tfrac{9}{12}) \\
&= 9 + (\tfrac{17}{12}) \\
&= 9 + (\tfrac{12}{12} + \tfrac{5}{12}) \\
&= 9 + (1 + \tfrac{5}{12}) \\
&= (9+1) + \tfrac{5}{12} \\
&= 10 + \tfrac{5}{12}\ (= 10\tfrac{5}{12})
\end{aligned}
$$

***Notes*:**

• We would not write the answer as $9\frac{17}{12}$. Remember that to be a mixed number, the fractional part has to be less than 1.

• Notice that because the denominator was 12 we traded 12 twelfths for 1 whole rather ten twelfths. Very often students make the mistake of not looking at the denominator of a mixed number and continue to "trade in" by tens no matter what the denominator was.

• We could also solve this problem by converting the two mixed numbers to improper fractions and then using our algorithm for adding fractions. This is illustrated below: However, the process is both more cumbersome and more "mechanical"

$$6\frac{2}{3} + 3\frac{3}{4} = \frac{20}{3} + \frac{15}{4}$$

$$= \frac{80}{12} + \frac{45}{12})$$

$$= \frac{80 + 45}{12}$$

$$= \frac{125}{12}$$

$$= 125 \div 12$$

$$= 10 \text{ R5}$$

$$= 10\frac{5}{12}$$

2. Subtracting Mixed Numbers

The definition of subtraction as "unadding" remains the same. That is, when we say that

$f - s$ means the number that we must add to s to obtain f as the sum

it does not matter whether f and s are whole numbers or rational numbers.

Practice Problem #3

Express $6\frac{7}{9} - 4\frac{2}{9}$ as a mixed number.

Answer: $2\frac{5}{9}$

Solution:

Since subtraction is "unadding" and since we add mixed numbers by adding the whole numbers and adding the fractions, we want to solve the following two problems:

$$4 + \_\_\_\_\_ = 6$$

and

$$\frac{2}{9} + \_\_\_\_\_ = \frac{7}{9}$$

Since $4 + 2 = 6$ and $\frac{2}{9} + \frac{5}{9} = \frac{7}{9}$, we see that the answer is $2\frac{5}{9}$.

As a quick check we see that $4\frac{2}{9} + 2\frac{5}{9} = 6\frac{7}{9}$.

Notes:

• We could have converted the mixed numbers to improper fractions by writing $6\frac{7}{9} - 4\frac{2}{9} = \frac{61}{9} - \frac{38}{9} = \frac{23}{9} = 2\frac{5}{9}$ [3]

• The only complication that might arise when we subtract mixed numbers is when we have to "borrow". As we mentioned in a previous note the idea of exchanging ten of one denomination for one of the next higher denomination is valid only when the denominator (if there is one) is 10. This illustrated in the next problem.

[3] In terms of our adjective/noun model, $7 - 2 = 5$ whenever 7, 2 and 5 modify the same noun. Hence: 7 ninths $-$ 2 ninths $=$ 5 ninths

Practice Problem #4

Express the difference $6\frac{2}{3} - 4\frac{3}{4}$ as a mixed number.

Answer: **$1\frac{11}{12}$**

Solution*:*

In terms of "unadding", the problem is asking us to fill in the blank if

$$4\frac{3}{4} + \_\_\_\_\_ = 6\frac{2}{3}$$

We begin as we did in the previous problem by filling in the blanks in the two statements

$$4 + \_\_\_\_\_ = 6 \qquad (1)$$

and

$$\frac{3}{4} + \_\_\_\_\_ = \frac{2}{3} \qquad (2)$$

We can only add or subtract fractions if they have the same dominator. Therefore we rewrite the above equation in the form:

$$\frac{9}{12} + \_\_\_\_\_ = \frac{8}{12}$$

The fact that $\frac{9}{12}$ is greater than $\frac{8}{12}$ means that we will have to "borrow" in order to solve the problem. This in turn means that we will rewrite $6\frac{2}{3}$ in the form:

$$\begin{aligned} 6\frac{2}{3} &= (5+1) + \frac{2}{3} \\ &= 5 + (1 + \frac{2}{3}) \\ &= 5 + (\frac{12}{12} + \frac{8}{12}) \\ &= 5 + \frac{20}{12} \\ &= 5\frac{20}{12} \end{aligned}$$

In a similar way, we see that:

$$4\frac{3}{4} = 4 + \frac{3}{4}$$
$$= 4 + \frac{9}{12}$$
$$= 4\frac{9}{12}$$

In other words, we may rewrite $6\frac{2}{3} - 4\frac{3}{4}$ in the equivalent form:

$$5\frac{20}{12} - 4\frac{9}{12}$$

Since $5 - 4 = 1$ and $\frac{20}{12} - \frac{9}{12} = \frac{11}{12}$, we see that

$$5\frac{20}{12} - 4\frac{9}{12} = 1\frac{11}{12}$$

<u>Notes</u>:

• To emphasize our "corn bread" model, we may think of our corn bread as being pre-sliced into 12 pieces of equal size. In this context we are saying that $5\frac{20}{12} - 4\frac{9}{12}$ represents the answer to the following problem.

You have 5 corn breads plus an additional 20 pieces of corn bread. You give away 11 of the 20 pieces and 4 of the 5 corn breads. How much corn bread do you have left?

Clearly you would have 1 corn bread left plus 9 pieces of the corn bread.

• The most common mistake that students are prone to make is that when they want to "borrow" 1 from $5\frac{8}{12}$, they will write $5\frac{{}^{1}8}{12}$ rather than $5\frac{{}^{2}0}{12}$.

When borrowing using mixed numbers, the new numerator is the *sum of the original numerator and the denominator.*

-- This is analogous to the fact that if you have 8 doughnuts and you buy another dozen, you have 20 doughnuts; not 18![4]

[4] It's usually easier for people to visualize things in terms of money. So, for example, if you have 3 quarters and you exchange a dollar for quarters, you now have 7 quarters, not 13 quarters! In more mathematical terms, $4 \times \frac{1}{4}$ (of a dollar) $= 1$ (dollar).

While converting mixed numbers to improper fractions is usually a more tedious way to subtract mixed numbers, it does avoid the "borrowing" problem. For example, we may rewrite $6\frac{2}{3}$ and $3\frac{3}{4}$ as:

$$
\begin{aligned}
6\tfrac{2}{3} &= 6 + \tfrac{2}{3} \\
&= \tfrac{6}{1} + \tfrac{2}{3} \\
&= \tfrac{18}{3} + \tfrac{2}{3} \\
&= \tfrac{20}{3} \\
&= \tfrac{80}{12}
\end{aligned}
\quad \text{and} \quad
\begin{aligned}
3\tfrac{3}{4} &= 3 + \tfrac{3}{4} \\
&= \tfrac{3}{1} + \tfrac{3}{4} \\
&= \tfrac{12}{4} + \tfrac{3}{4} \\
&= \tfrac{15}{4} \\
&= \tfrac{45}{12}
\end{aligned}
$$

Hence $6\frac{2}{3} - 3\frac{3}{4} = \frac{80}{12} - \frac{45}{12} = \frac{35}{12} = 2\frac{11}{12}$

This concludes part 2 of this lesson. in part 3 we will discuss how we multiply mixed numbers.