Unedited Draft

Arithmetic Revisited

Lesson 4:

Part 3: Multiplying Mixed Numbers

1. Introduction:

As we mentioned in a note on the section on adding mixed numbers, because the plus sign is "missing", it is easy to overlook the fact that when we multiply, say, $2\frac{1}{2}$ by $4\frac{1}{2}$, we must use the distributive property. For example, there is a tendency by students to multiply the two whole numbers to get 8 and the two fractions to get $\frac{1}{4}$. From this they conclude that $2\frac{1}{2} \times 4\frac{1}{2} = 8\frac{1}{2}$. However, since 2 is less than $2\frac{1}{2}$, we know that $2 \times 4\frac{1}{2}$ is less than $2\frac{1}{2} \times 4\frac{1}{2}$; and since $2 \times 4\frac{1}{2} = 9$, we know that $2\frac{1}{2} \times 4\frac{1}{2}$ must be greater than 9. Hence it is impossible for the correct answer to be $8\frac{1}{2}$. [1]

Perhaps this will be easier to understand if we look at a real-life illustration

Practice Problem #1

How much will you have to pay for candy in order to buy $2\frac{1}{2}$ pounds if the candy costs $4.50 per pound?

Answer $11.25

Solution:

Stated in its present form, the problem is a simple arithmetic problem. Namely, at $4.50 per pound, 2 pounds would cost $9 and a half pound would cost half of $4.50 or $2.25. Hence the total cost is $11.25.

Notes:

- **What we have done is multiplied $2\frac{1}{2}$ pounds by $4\frac{1}{2}$ dollars and obtained $11\frac{1}{4}$ dollars as the product. This validates our observation that the product of $2\frac{1}{2}$ and $4\frac{1}{2}$ is greater than 9.**

- **In fact, not only does our result validate that the product is greater than 9 but it also tells us that $4\frac{1}{2} \times 2\frac{1}{2} = 11\frac{1}{4}$.**

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[1]This is a point that is well worth emphasizing. Namely, even when we don't know the correct answer it is important to have enough number sense to be able to decide when an answer can't be correct.

• Notice that if you were the store owner and believed that $2\frac{1}{2} \times \$4\frac{1}{2} = \$8\frac{1}{4}$, you would have "short changed" yourself by \$3 on this transaction. So even if it does seem "natural" or "logical" to multiply the whole numbers and multiply the fractions, it just doesn't work in the real world.

• So whether it is more difficult to multiply the "correct" way is not the issue. The issue is that if we want to multiply mixed numbers, we have to pay attention to the distributive property. More specifically:

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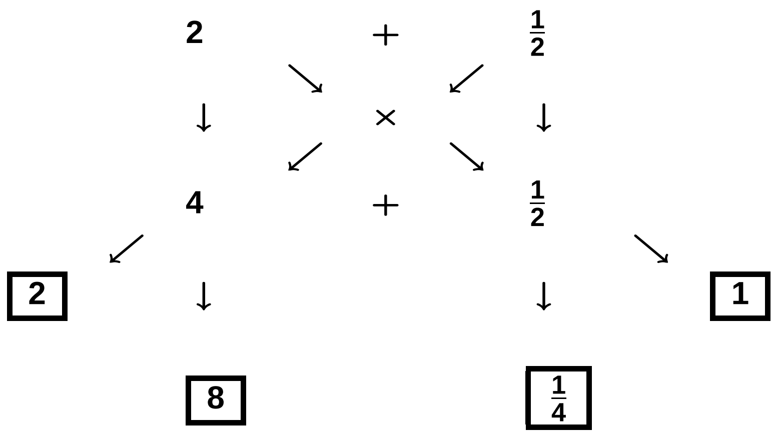
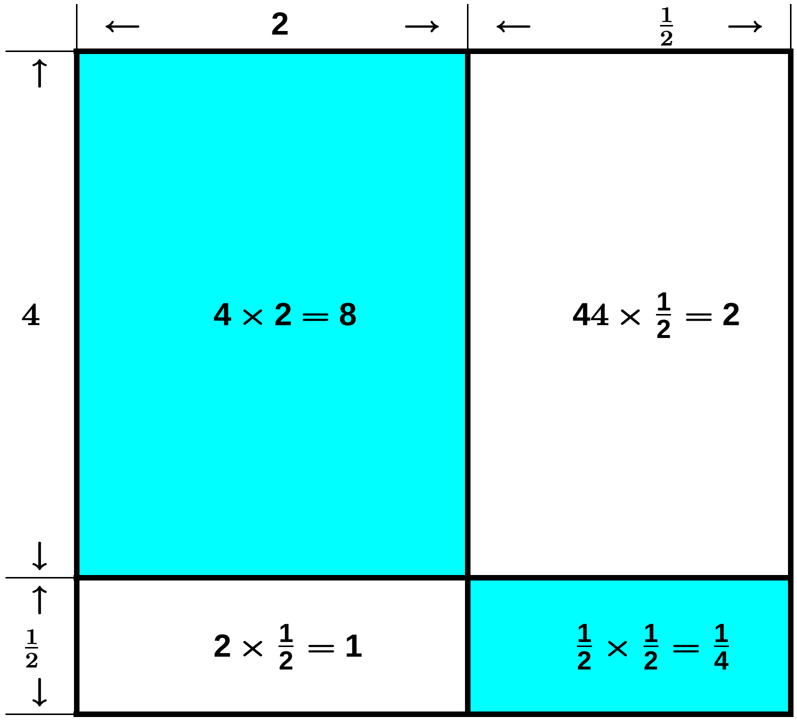
$$\begin{aligned}(2 + \tfrac{1}{2}) \times (4 + \tfrac{1}{2}) &= (2 \times 4) + (2 \times \tfrac{1}{2}) + (\tfrac{1}{2} \times 4) + (\tfrac{1}{2} \times \tfrac{1}{2}) \\ &= 8 + 1 + 2 + \tfrac{1}{4} \\ &= 11\tfrac{1}{4}\end{aligned}$$

• In schematic form we may represent the distributive property as follows:



2. Using the Area Model:

For students who are more visually oriented, we may use the area model to illustrate the distributive property (just as we did in our discussion of whole number multiplication). More specifically:



- **The "big" rectangle has dimensions $4\frac{1}{2}$ by $2\frac{1}{2}$**
- **Its area is the sum of the areas of the four smaller rectangles inside. That is, the total area is $4 + 1 + 2 + \frac{1}{4}. = 11\frac{1}{4}$**
- **The shaded region represents the region that is represented by what happens when we "multiply the whole numbers and we multiply the two fractions".**

3. Converting the Mixed Numbers to Improper Fractions

If we prefer not to use the distributive property, we may convert both mixed numbers to improper fractions and solve the problem that way. In other words

$$\begin{aligned} 4\tfrac{1}{2} \times 2\tfrac{1}{2} &= \tfrac{9}{2} \times \tfrac{5}{2} \\ &= \tfrac{9 \times 5}{2 \times 2} \\ &= \tfrac{45}{4} \\ &= 11\tfrac{1}{4} \end{aligned}$$

Practice Problem #2

Express the product $5\frac{3}{7} \times 7\frac{2}{5}$ as a mixed number

Answer: $40\frac{6}{35}$

Solution:

Using the distributive property, we see that:

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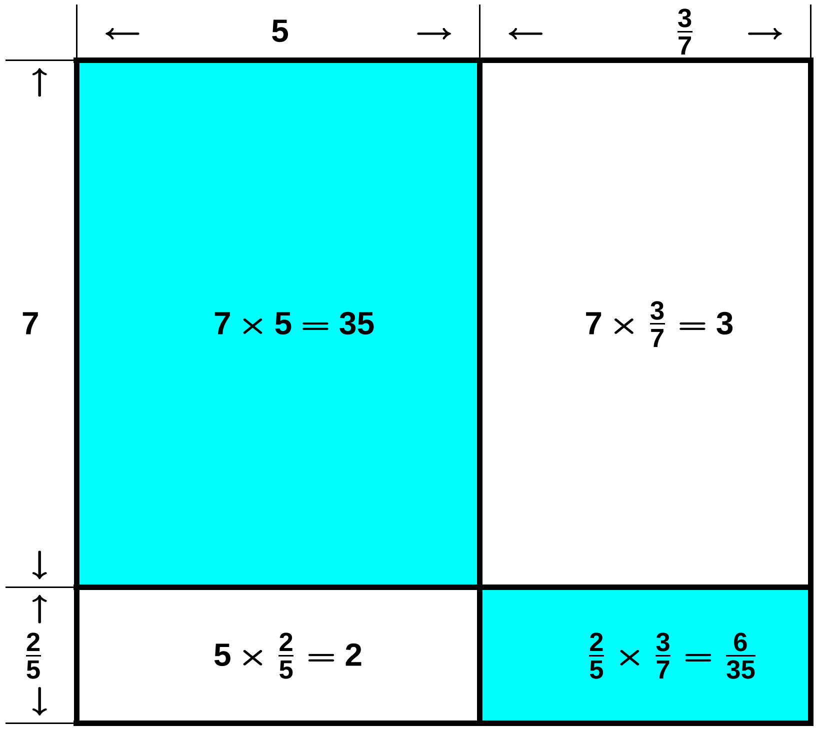
$$\begin{aligned} (5 + \tfrac{3}{7}) \times (7 + \tfrac{2}{5}) &= (5 \times 7) + (5 \times \tfrac{2}{5}) + (\tfrac{3}{7} \times 7) + (\tfrac{3}{7} \times \tfrac{2}{5}) \\ &= 35 + 2 + 3 + \tfrac{6}{35} \\ &= 40\tfrac{6}{35} \end{aligned}$$

***Notes*:**

- **For students who are more visually oriented, we may use the area model to illustrate the distributive property (just as we did in our discussion of whole number multiplication). More specifically:**



- **We could also solve this problem by converting the mixed numbers into improper fractions. Namely:**

$$\begin{aligned} 5\tfrac{3}{7} \times 7\tfrac{2}{5} &= \frac{38}{7} \times \frac{37}{5} \\ &= \frac{38 \times 37}{7 \times 5} \\ &= \frac{1{,}406}{35} \\ &= 40\tfrac{6}{35} \end{aligned}$$

- **To generalize the above result, the recipe for computing the product of two mixed numbers by using improper fractions is:**

- **Rewrite the mixed number problem as an equivalent improper fraction problem.**
- **Solve the resulting improper fraction problem.**
- **Translate the answer from an improper fraction into a mixed number**

Caution:

In finding the product of 37 and 38 it is permissible to use a calculator. However, even with a calculator one can make such careless errors as pressing an incorrect key. Therefore it is a good idea to estimate the answer even before you do the actual computation. In this case:

$$\begin{array}{ccccccc} & 5 & < & 5\frac{3}{7} & < & 6 \\ \times & 7 & < & 7\frac{2}{5} & < & 8 \\ \hline & 35 & < & ? & < & 48 \end{array}$$

Thus any answer that is less than 35 or greater than 48 must be incorrect

• Remember that the actual arithmetic involves only the adjectives. The noun that the answer modifies has to be determined by the actual problem. For example:

-- If the problem was to find the area of a rectangle whose length was $7\frac{2}{5}$ inches and whose width was $5\frac{3}{7}$ inches, the answer would be

$$7\frac{2}{5} \text{ inches} \times 5\frac{3}{7} \text{ inches} = (7\frac{2}{5} \times 5\frac{3}{7}) \text{ “inch-inches”}$$

$$= 40\frac{6}{35} \text{ inch}^2$$

$$= 40\frac{6}{35} \text{ square inches}$$

-- and if an object moved a constant speed of $7\frac{2}{5}$ miles per hour for $5\frac{3}{7}$ hours, it would have traveled $40\frac{6}{35}$ miles during this time

Summary:

• From a “mechanical” point of view, the “easiest” way to multiply mixed numbers is to convert them into improper fractions first and then multiply. in essence, this method reduces the problem of multiplying mixed numbers into an equivalent problem that involves multiplying (improper) fractions.

• However, if we convert the mixed numbers into improper fractions we do not see why the answer turns out the way it does. For this reason it is a good idea for students to understand that a mixed number is a sum; so that if we multiply them without first converting them to improper fractions, we have to use the distributive property.

- **However, even if students understand how to use the distributive property, they might still not be able to visualize what the process actually does. For this reason it is very nice for the students to understand the area model.**

In any event, this completes our discussion of how we multiply mixed numbers. In the next part of the lesson we will discuss how we divide mixed numbers