

****Unedited Draft****

Arithmetic Revisited

Lesson 5:

Decimal Fractions or Place Value Extended

Part 2: Adding and Subtracting Decimals

1. Adding Decimals

Decimal arithmetic is very similar to whole number arithmetic. More specifically, decimals preserve the properties of place value and once we know where to place the decimal point, the arithmetic algorithms remain exactly the same.¹ When we add or subtract whole numbers using place value, we line the numbers up in vertical form so that numbers in the same column modify the same noun. In doing this we knew that the digit furthest to the right in each summand modified the ones place. The only difference now is that the ones place is modified by the first digit to the left of the decimal point. Therefore, rather than to line the numbers up using the digit furthest to the right, we line the numbers up using the decimal point. Aligning the decimal points guarantees that numbers in the same column modify the same noun. By way of example:

Illustrative Example #1.

Find the sum of 28.3 and 7.93

Step 1:

Write the numbers in vertical form with the decimal points aligned

$$\begin{array}{r} 28.3 \\ + 7.93 \\ \hline \end{array}$$

Step 2:

“Bring down” the decimal point

$$\begin{array}{r} 28.3 \\ + 7.93 \\ \hline . \end{array}$$

¹In fact, the arithmetic is so similar that students (especially those who are willing to memorize rather than understand) often feel that decimals should have been taught before (or even in place of) common fractions. In fact, in recent years the stock market has decided to use decimal fractions rather than common fractions when talking about shares of stock. However, the concept of rational numbers requires that we understand common fractions, after which decimal fractions are a special case of sorts.

Step 3:

Then add as you would if there was no decimal point.

$$\begin{array}{r}
 28.3 \\
 + 7.93 \\
 \hline
 36.23
 \end{array}$$

Notes

- A mistake that some students tend to make can be nicely resolved in terms of the “adjective/noun” theme. More specifically, they are used to the fact that in dealing with whole numbers the alignment is based on the digits furthest to the right and hence they might write the “solution” as

$$\begin{array}{r}
 28.3 \\
 + 7.93 \\
 \hline
 10.76^2
 \end{array}$$

- The same type of mistake can happen even if students use calculators. Granted that the calculator keeps track of the decimal point automatically, it is possible that by mistake you entered 2.83 rather than 28.3, in which case the calculator would yield 10.76 as the answer.
- Thus, as “easy” as decimal arithmetic might seem, it is still important to have some sort of number sense, at least in the form of being able to arrive at a reasonable estimate for the answer. For example, we know that 28.3 is between 28 and 29 and that 7.93 is between 7 and 8. Hence the sum has to be between 35 and 37 (or more simply, rounded off to the nearest whole number 29.3 is 28 and 7.93 is 8. Hence the sum has to be approximately 28 + 8 or 36). That is:

$$\begin{array}{r}
 28 < 28.3 < 29 \\
 7 < 7.93 < 8 \\
 \hline
 35 < ? < 37
 \end{array}$$

whereupon it's easy to see that 10.76 can't possibly be the correct answer.

²The adjective/ noun theme tells us that when we write $3 + 3 = 6$, we are assuming that the numbers modify the same noun. However in 28.3 the 3 modifies tenths while in 7.93 the 3 modifies hundredths. We avoid this error by aligning the decimal points before we add.

- Since 28.3 and 28.30³ mean the same thing, we may replace 28.3 by 28.30 to “even off” the columns. That is, to make it look more like a whole number problem, we could have written the problem as

$$\begin{array}{r} 28.30 \\ + 7.93 \\ \hline 36.23 \end{array}$$

- To see how the above vertical form is related to our adjective/noun theme, notice that we may read 28.30 as 2,830 *hundredths*; and that we may read 7.93 as 793 *hundredths*. Hence what we were really doing is the *whole number* problem:

$$\begin{array}{r} 2,830 \text{ hundredths} \\ + 793 \text{ hundredths} \\ \hline 3,623 \text{ hundredths} \end{array}$$

and we “get rid of” the word “hundredths” by placing the decimal point two places to the left of the digit that is furthest to the right.

Practice Problem #1

Express the sum $23.97 + 147.6 + 1.368$ as a decimal fraction.

Answer: 172.938

Solution:

In vertical format, we arrange the three numbers so that their decimal points are aligned. We then make sure that the decimal point in the answer is aligned with those above. That is:

$$\begin{array}{r} 23.97 \\ + 147.6 \\ + 1.368 \\ \hline 172.938 \end{array}$$

³The zero in 28.30 does nothing to change the denominations that are modified by the 2, 3, or 8. The same result can be seen as application of common fractions. Namely 28.3 means $\frac{283}{10}$ and if we multiply numerator and denominator by 10 we obtain the equivalent common fractions $\frac{2830}{100}$ which means the same thing as 2.830. However there is a difference in terms of significant figures. Namely 28.3 indicates that the measurement is accurate only to the nearest tenth, while 28.30 tells us that the measurement is accurate to the nearest hundredth.

If the columns look “unbalanced” rewrite the above in the form⁴

$$\begin{array}{r}
 23 \bullet 970 \\
 + 147 \bullet 600 \\
 + \quad 1 \bullet 368 \\
 \hline
 \bullet
 \end{array}$$

Then add in the “usual way” as if there were no decimal points

$$\begin{array}{r}
 23 \bullet 970 \\
 + 147 \bullet 600 \\
 + \quad 1 \bullet 368 \\
 \hline
 172 \bullet 938
 \end{array}$$

Note:

By aligning the decimal points in the above form, we see that the digit furthest to the right in all three numbers is modifying thousandths. Hence the above form is merely a different “language” for writing

$$\begin{array}{r}
 23,970 \text{ thousandths} \\
 + 147,600 \text{ thousandths} \\
 + 1,368 \text{ thousandths} \\
 \hline
 172,938 \text{ thousandths}
 \end{array}$$

Whenever we introduce the concept of addition we know that the concept of “unadding”, or subtraction, cannot be far behind. With this in mind we now turn our attention to subtracting decimals.

⁴Remember that putting in 0's doesn't change anything unless the 0 moves a non zero digit to a new denomination. thus $23.97 = 23.970$ but $23.97 \neq 23.097$.

Illustrative Example #2:

Subtract 7.93 from 28.3

Answer: 20.37

Solution:

Except for the minus sign replacing the plus sign, the solution looks exactly the same as in Illustrative Example #1. Namely, starting with

$$\begin{array}{r} 28.30 \\ - 7.93 \\ \hline \end{array}$$

we subtract as if we were dealing with whole numbers to obtain:

$$\begin{array}{r} 28.30 \\ - 7.93 \\ \hline 20.37 \end{array}$$

Notes:

- Notice the importance of the 0 in this example. Students often tend to think of 0 as meaning “nothing” and when they write the problem as

$$\begin{array}{r} 28.3 \\ - 7.93 \\ \hline \end{array}$$

there is a danger that they will forget that they are subtracting 3 from “0”. Rather they simply ignore the 0 and “bring down” the 3⁵. In other words, there is a tendency for them to write:

$$\begin{array}{r} 28.3 \\ - 7.93 \\ \hline 20.43^6 \end{array}$$

⁵In addition when we omit the 0 we are merely restating that $3 + 0 = 3$. That is, adding “nothing” to a number doesn't change the number. However subtracting a number from “nothing” isn't the same as doing nothing to the number. In symbols, $3 + 0 = 3$ but $0 - 3 \neq 3$.

⁶This error does not occur when we use a calculator because the calculator is programmed to keep track of each place. Nevertheless, it is important for students to understand each process and why it works (especially if they don't happen to have their calculators with them!).

Illustrative Example #3

Express the difference $12.346 - 8.7684$ as a decimal fraction.

Answer: 4.5776

Solution:

Just as in the case of addition, we begin by writing the problem in vertical form with the decimal points aligned.

$$\begin{array}{r} 12.346 \\ - 8.7684 \\ \hline \end{array}$$

Again, to “balance” the columns, we can use a zero and rewrite the above as

$$\begin{array}{r} 12.3460 \\ - 8.7684 \\ \hline \end{array}$$

We then ignore the decimal points and subtract as if we were dealing with whole numbers. That is:

$$\begin{array}{r} 123460 \\ - 87684 \\ \hline 45776 \end{array}$$

Notes:

- As usual we can check subtraction by addition. That is, if we have subtracted correctly it means that $4.5776 + 8.7684 = 12.346$; and this is verified as shown below:

$$\begin{array}{r}
 1 \quad 2 \quad \bullet \quad 3 \quad 4 \quad 6 \quad 0 \\
 - \quad 8 \quad \bullet \quad 7 \quad 6 \quad 8 \quad 4 \\
 \hline
 \quad 4 \quad \bullet \quad 5 \quad 7 \quad 7 \quad 6 \\
 + \quad 8 \quad \bullet \quad 7 \quad 6 \quad 8 \quad 4 \\
 \hline
 1 \quad 2 \quad \bullet \quad 3 \quad 4 \quad 6 \quad 0
 \end{array}$$

- If the 0 had been omitted and we subtracted by “bring down” the 4 as shown below

$$\begin{array}{r}
 1 \quad 2 \quad \bullet \quad 3 \quad 4 \quad 6 \\
 - \quad 8 \quad \bullet \quad 7 \quad 6 \quad 8 \quad 4 \\
 \hline
 \quad 4 \quad \bullet \quad 5 \quad 7 \quad 8 \quad 4
 \end{array}$$

we would get an incorrect result if we checked our answer by addition. That is:

$$\begin{array}{r}
 1 \quad 2 \quad \bullet \quad 3 \quad 4 \quad 6 \\
 - \quad 8 \quad \bullet \quad 7 \quad 6 \quad 8 \quad 4 \\
 \hline
 \quad 4 \quad \bullet \quad 5 \quad 7 \quad 8 \quad 4 \\
 + \quad 8 \quad \bullet \quad 7 \quad 6 \quad 8 \quad 4 \\
 \hline
 1 \quad 2 \quad \bullet \quad 3 \quad 4 \quad 6 \quad 8
 \end{array}$$

- In terms of having whole number adjectives, the above method is just a different form of writing

$$\begin{array}{r}
 123,460 \quad \text{ten-thousandths} \\
 - \quad 87,684 \quad \text{ten-thousandths} \\
 \hline
 45,784 \quad \text{ten-thousandths}
 \end{array}$$

What the above examples show us is that as far as addition and subtraction are concerned, the arithmetic of decimal fractions is “almost the same” as how we use place value in doing the arithmetic of whole numbers. In particular, by choosing the correct denominations, every decimal problem can be translated into an equivalent whole number problem. The same idea applies to the product of decimals; and this will be the subject of the next part of this lesson.