Arithmetic Revisited

Lesson 5:

Decimal Fractions or Place Value Extended

Part 6: Dividing Decimal Fractions, Part 3

Enrichment: Converting Repeating Decimals into Common Fractions

The fact that every rational number can be represented by either a decimal that terminates or else by a decimal that eventually repeats the same cycle of digits endlessly does not mean in itself that the "reverse" (called the *converse*) is true. In this respect, so far we have shown that every terminating decimal can be represented as a common fraction. However, it still remains to be proven that *any* decimal that *eventually repeats the same cycle of digits* endlessly *must* represent a rational number.[1]

Demonstrating this is a bit tricky in the sense that it is easier to scramble an egg than to unscramble one. For example, if we start with $1 \div 3$ and want to write it in decimal form, we simply go through the division process and see that the decimal is $0.\overline{3}$. However, suppose that we didn't already know that $0.\overline{3}$ was equal to $\frac{1}{3}$ and we wanted to express it as an equivalent common fraction, where would we begin?[2]

As we've mentioned before, from a practical point of view there is no need to know how to do this because we can get as an exact approximation as is needed in a "real world" application. However there is a side of mathematics that belongs as much to the humanities as it does to science and technology. In fact, there are people who study mathematics for the same reason that there are people who study poetry; not to build better bridges but because it represents a certain type of beauty; a beauty that shows the heights to which the human mind can soar. In this context, it would simply be "nice" to know that the converse is true. In fact, once we succeed in proving this we will have developed a very strong connection between rational numbers and decimals that may help us understand what irrational numbers (which will be discussed in the next part of this lesson)are and how abundantly they exist.

Having said all of this, let's now try to see how we could have started with $0.\overline{3}$ and determined that it represented $\frac{1}{3}$.

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[1]An easy to visualize why the converse of a true statement need not be true is to consider the fact that while it is true that every bear is an animal, it is not true that every animal is a bear.

[2]Unlike in what happens when we have a terminating decimal, in the case of $0.\overline{3}$ (or in the case of any non-terminating decimal) *there is no digit "furthest to the right"*. In other words, the definition of *endless* means that we *never* come to the *end* of the decimal.

- **To begin with, since the repeating cycle in $0.\overline{3}$ consists of just a single digit (3), if we move the decimal point just 1 place to the right, the part that now follows the decimal point is still $.\overline{3}$**

- **However, if we move the decimal point 1 place to the right, we have multiplied the decimal by 10. In other words, if we let n stand for $0.\overline{3}$, then 10n stands for $3.\overline{3}$ In other words,**

$$10n = 3.\overline{3}$$

and

$$n = 0.\overline{3}$$

The strategy we will now use relies on the fact that any number subtracted from itself is 0 (*even if the number is represented by an endless decimal*).

So let's subtract the bottom line above from the top line above. Recall that n means the same thing as 1n. Hence the left hand side becomes $10n - 1n$ or 9n; and on the right hand side the $.\overline{3}$ in the bottom line "cancels" with the $.\overline{3}$ in top line. That is, $3.\overline{3} - 0.\overline{3} = 3$. In summary:

$$\begin{array}{rrcl} & 10n & = & 3.\overline{3} \\ - & 1n & = & 0.\overline{3} \\ \hline & 9n & = & 3.0 \end{array}$$

And if $9n = 3$, then $n = 3 \div 9 = \frac{3}{9} = \frac{1}{3}$.

> **In summary, to convert a non-terminating repeating decimal into a common fraction, move the decimal point in such a way that we get *two* decimals that have the *same* fractional part (that is, the same repeating cycle). We then *subtract* the *lesser* decimal *from* the *greater* to get the relationship that yields the desired fraction.**

Rather than belabor this point, let's try a couple of examples on our own.

<u>Illustrative Example:</u>

Write each of the following as an equivalent common fraction in lowest terms:

(a) 0.216

(b) $0.\overline{216}$

(c) $0.21\overline{6}$

Solving part (a) is relatively easy. Namely, 0.216 terminates and the digit furthest to the right is 6. Thus the numerator of an equivalent common fraction will be 216; and since there are three digits (216) to the right of the decimal point, the denominator of that common fraction will be 1,000.

Hence one equivalent common fraction is $\frac{216}{1,000}$**; but since the numerator and denominator of this fraction are both divisible by 8, the fraction can written in reduced form as** $\frac{27}{125}$**. As a check we see that:**

$$
\begin{array}{r|cccccc}
 & & 0. & 2 & 1 & 6 \\
\hline
125 & 2 & 7. & 0 & 0 & 0 \\
- & 2 & 5 & 0 & & \\
\cline{2-5}
 & & 2 & 0 & 0 & \\
- - & & 1 & 2 & 5 & \\
\cline{4-5}
 & & & 7 & 5 & 0 \\
 & & - & 7 & 5 & 0 \\
\cline{5-6}
 & & & & & 0
\end{array}
$$

However, we cannot use the same strategy for doing part (b) because $0.\overline{216}$ **has no "last" digit. However, it repeats the cycle of digits 216 endlessly. Thus, the same cycle will repeat if we move the decimal point 3 places to the right. Moving the decimal point 3 places to the right is equivalent to multiplying the original decimal by 1,000.**

Hence if we now let n denote the original decimal we have:

$$
\begin{array}{rcr}
1{,}000n & = & 216.\overline{216} \\
-\quad 1n & = & -\quad 0.\overline{216} \\
\hline
999n & = & 216
\end{array}
$$

The fact that $999n = 216$ **means that** $n = \frac{216}{999}$**, and if we reduce this to lowest terms (27 is a common factor of the numerator and denominator), we see that** $n = \frac{8}{37}$**.**

As a check we see that:

		0.	2	1	6
37	)	8.	0	0	0
	– –	7	4		
			6	0	
		–	3	7	
			2	3	0
		–	2	2	2
					8[3]

Caution:

There is a tendency for students to ignore the bar in $0.\overline{216}$ and write 0.216 instead of $0.\overline{216}$. While this may not produce a "significant" error; it is, nevertheless, incorrect to ignore the bar.

Part (c) is bit "trickier" because the repeating cycle consists only of the "6". The first place we can put the decimal point so that the fractional part is $.\overline{6}$ is after the "1". That is, if we move the decimal point two places to the right we get the number $21.\overline{6}$. Moving the decimal point two places to the right is the same as multiplying the original decimal by 100.

The next place we could move the decimal point so that the fractional part is still $.\overline{6}$ is after the first "6". That is, if we move the decimal point three places to the right we get the number $216.\overline{6}$. Moving the decimal point three places to the right is the same thing as multiplying the original decimal by 1,000.

Therefore, if now let m denote the original decimal, we have that

$$\begin{array}{rcl} 1{,}000m & = & 216.\overline{6} \\ -\ 100m & = & 21.\overline{6} \\ \hline 900m & = & 195 \end{array}$$

The fact that $900m = 125$ tells us that $m = \frac{195}{900}$ and dividing numerator and denominator by 15 we see that in lowest terms $m = \frac{13}{60}$. as a check we see that

[3]We would have obtained the same result if we used the division algorithm to express 216 ÷ 999 as a common fraction. However by using 8 ÷ 27 we were able to work with less cumbersome numbers. Notice that as far as the decimal representation is concerned it doesn't matter whether we use 216 ÷ 999 or 8 ÷ 37.

		0.	2	1	6
60	1	3.	0	0	0
– –	1	2	0		
		1	0	0	
–			6	0	
			4	0	0
–			3	6	0
				4	0[4]

<u>Commentary</u>:

• Notice the importance of where we place the"bar". In this problem, all three parts look like 0.216 without the "bar"; yet all three parts are different. In particular, don't confuse 0.216 and $0.\overline{216}$. In fraction form, $0.216 = \frac{216}{1,000}$ while $0.\overline{216} = \frac{216}{999}$ and $0.21\overline{6} = \frac{13}{60}$.

• The letter that we use to denote the decimal is irrelevant; that is, the letter simply takes the place of a "blank". However, it is a generally accepted agreement in mathematics that we can not use the same letter for two different amounts in a problem. Therefore, to make sure there would be no confusion, once we let n stand for $0.\overline{216}$, we had to let a different letter (we chose m) to stand for $0.21\overline{6}$.

<u>Practice Problem #1</u>

Write each of the following as a common fraction in lowest terms.

(a) $0.\overline{432}$ (b) $0.43\overline{2}$

Answers: (a) $\frac{16}{37}$ (b) $\frac{389}{900}$

<u>Solution</u>*:*

<u>Part (a)</u>

$0.\overline{432}$ has no "last" digit. However, it repeats the cycle of digits 432 endlessly. Thus, the same cycle will repeat if we move the decimal point 3 places to the right. Moving the decimal point 3 places to the right is equivalent to multiplying the original decimal by 1,000.

Hence if we now let nn denote the original decimal we have:

$$\begin{array}{rcl} 1{,}000n & = & 432.\overline{432} \\ -\ 1n & = & 0.\overline{432} \\ \hline 999n & = & 432. \end{array}$$

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[4]Notice that the first repeated remainder (40) was not the first remainder (10). that's why the repetition did not begin with the first digit to the right of the decimal point.

The fact that $999n = 432$ means that $n = \frac{432}{999}$, and if we reduce this to lowest terms (27 is a common factor of numerator and denominator), we see that $n = \frac{16}{37}$.

We can check the result by using the division algorithm to compute $16 \div 37$. the division yields:

$$\begin{array}{r|rrrrrrrr} & & 0. & 4 & 3 & 2 & & & \\ \hline 37 & 1 & 6. & 0 & 0 & 0 & 0 & 0 & 0 \\ -- & 1 & 4 & 8 & & & & & \\ \hline & & 1 & 2 & 0 & & & & \\ -- & & 1 & 1 & 1 & & & & \\ \hline & & & & 9 & 0 & & & \\ -- & & & & 7 & 4 & & & \\ \hline & & & & 1 & 6 & & & \end{array}$$

We began the division with the dividend being 16 followed only by 0's. In the last line of the algorithm we again have 16 followed only by 0's Hence the sequence of digits 432 will repeat endlessly.

<u>Part (b)</u>

In this part, the repeating cycle consists only of the "2" In other words, the 4 and the 3 are not part of the repeating cycle. The first place we can put the decimal point so that the fractional part is $.\overline{2}$ is after the "3". That is, if we move the decimal point two places to the right we get the number $43.\overline{2}$. Moving the decimal point two places to the right is the same as multiplying the original decimal by 100.

The next place we could move the decimal point so that the fractional part is still $.\overline{2}$ is after the first "2". That is, if we move the decimal point three places to the right we get the number $432.\overline{2}$. Moving the decimal point three places to the right is the same thing as multiplying the original decimal by 1,000.

Therefore, if now let m denote the original decimal, we have that

$$\begin{array}{rcr} 1{,}000m & = & 432.\overline{2} \\ -\ \ 100m & = & 43.\overline{2} \\ \hline 900m & = & 389.0 \end{array}$$

The fact that $900m = 389$ tells us that $m = \frac{389}{900}$. The only prime factors of 900 are 2, 3 and 5.. However, none of these is a factor of 389. Hence $\frac{389}{900}$ is in lowest terms.

As a check, we see that:

			0.	4	3	2	2	2	2
900	3	8	9.	0	0	0	0	0	0
— —	3	6	0	0					
		2	9	0	0				
	— —	2	7	0	0				
			2	0	0	0			
		— —	1	8	0	0			
				2	0	0	0		
			— —	1	8	0	0		
					2	0	0	etc.	

This concludes this part of the lesson. in the next part we will discuss what happens if there is a non-terminating decimal that doesn't eventually repeat the same cycle of digits endlessly.

A Major Point

The decimal representation of any rational number either is a terminating decimal or else eventually repeats the same cycle of digits endlessly.

Stated a bit differently, the decimal equivalent of any common fraction either terminates or else eventually repeats the same cycle of digits endlessly.

The decimal equivalent of a common fraction that is represented in lowest terms will terminate if and only if the only prime factors of the denominator of the fraction are 2's and/or 5's.